F-theory duals of M-theory on G_2 manifolds from mirror symmetry

2003 J. Phys. A: Math. Gen. 36 4191

(http://iopscience.iop.org/0305-4470/36/14/319)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4191–4206 PII: S0305-4470(03)52048-0

F-theory duals of M-theory on G_2 manifolds from mirror symmetry

Adil Belhaj

High Energy Physics Laboratory, Physics Department, Faculty of Sciences, Avenue Ibn Battouta, PO Box 1014, Rabat, Morocco

E-mail: ufrhep@fsr.ac.ma

Received 2 August 2002, in final form 12 February 2003
Published 26 March 2003
Online at stacks.iop.org/JPhysA/36/4191

Abstract

Using mirror pairs (M_3, W_3) in type II superstring compactifications on Calabi–Yau threefolds, we study, geometrically, F-theory duals of M-theory on seven manifolds with G_2 holonomy. We first develop a way of obtaining Landau–Ginzburg (LG) Calabi–Yau threefolds W_3, embedded in four complex-dimensional toric varieties, mirror to the sigma model on toric Calabi–Yau threefolds M_3. This method gives directly the right dimension without introducing non-dynamical variables. Then, using toric geometry tools, we discuss the duality between M-theory on $\frac{S^1 \times M_3}{\mathbf{Z}_2}$ with G_2 holonomy and F-theory on elliptically fibred Calabi–Yau fourfolds with $SU(4)$ holonomy, containing W_3 mirror manifolds. Illustrative examples are presented.

PACS numbers: 11.25.Yb, 02.40.Tt

1. Introduction

Over the past few years, there has been increasing interest in studying string dualities. This interest is due to the fact that this subject allows us to explore several connections between different models in string theory. Interesting examples are mirror symmetry between pairs of Calabi–Yau manifolds in type II superstrings [1–4] and strong/weak coupling duality among this type and heterotic superstrings on $K3 \times T^2$ [5]. The most important consequence of the study of string duality is that all superstring models are equivalent in the sense that they correspond to different limits in the moduli space of the same theory, called M-theory [6–9]. The latter, which is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models, is described, at low energies, by an eleven-dimensional supergravity theory.

More recently, special interest has been drawn to the study of the compactification of the M-theory on seven real manifolds X_7 with non-trivial holonomy providing a potential point of contact with low energy semirealistic physics in our world [9, 10]. In particular, one can obtain four-dimensional theory with $N = 1$ supersymmetry by compactifying M-theory on $R^{1,3} \times X_7$ where X_7 is a seven manifold with G_2 exceptional holonomy [10–23]. This result can be understood by the fact that the G_2 group is the maximal subgroup of $SO(7)$ for which a eight-dimensional spinor of it can be decomposed as a fundamental of G_2 and one singlet. In this regard, the $N = 1$ four-dimensional resulting physics depends on geometric properties of X_7. For instance, if X_7 is smooth, the low energy theory contains, in addition to $N = 1$ supergravity, only Abelian gauge group and neutral chiral multiplets. In particular, one has $b_2(X_7)$ Abelian vector multiplets and $b_3(X_7)$ massless neutral chiral multiplets, b_i denote the Betti numbers of X_7. The non-Abelian gauge symmetries with chiral fermions can be obtained by considering limits where X_7 develops singularities [17, 18, 21]. The $N = 1$ theory in four dimensions can also be obtained using alternative ways. One way is to consider the $E_8 \times E_8$ heterotic superstring compactifications. In this way, the compact manifold is a Calabi–Yau threefold with $SU(3)$ holonomy with an appropriate choice of vector bundle over it breaking the $E_8 \times E_8$ gauge symmetry [24].

Another way, which is dual to the heterotic compactification, is to compactify F-theory on elliptically fibred Calabi–Yau fourfolds with $SU(4)$ holonomy group [25–28]. At this level, one might naturally ask the following questions. Is there a duality between M-theory on seven G_2 manifolds and F-theory on Calabi–Yau fourfolds and what will be the geometries behind this duality?

In this paper, we address these questions using toric geometry and mirror pairs in type II superstrings propagating on Calabi–Yau threefolds. In particular, we discuss the duality between M-theory on G_2 manifolds and F-theory on elliptically fibred Calabi–Yau fourfolds with $SU(4)$ holonomy. In this study, we consider M-theory on spaces of the form $\frac{S^1 \times M_3}{\mathbf{Z}_2}$ with G_2 holonomy where M_3 is a local Calabi–Yau threefold described physically by $N = 2$ sigma model in two dimensions. The F-theory duals of such models can be obtained by the compactification on $\frac{T^2 \times W_3}{\mathbf{Z}_2}$, where (M_3, W_3) are mirror pairs in type II superstring compactifications. More precisely, using toric geometry tools, we first develop a way of obtaining LG Calabi–Yau threefolds W_3 mirror to the sigma model on toric Calabi–Yau M_3. This method is based on solving the mirror constraint equations for LG theories in terms of the toric data of the sigma model on M_3. Then we give a toric description of the above-mentioned duality. In particular, we propose a special $\mathbf{Z}_2$ symmetry acting on the toric geometry angular variables producing seven manifolds with $K3$ fibrations in G_2 manifold compactifications.

This paper is organized as follows. In section 2, using mirror symmetry in type II superstrings, we propose a possible duality between M-theory on G_2 manifolds and F-theory on elliptically fibred Calabi–Yau fourfolds involving mirror pairs of Calabi–Yau threefolds (M_3, W_3). In section 3 we develop a method for obtaining LG Calabi–Yau threefolds W_3 mirror to the sigma model on toric Calabi–Yau M_3. This method is based on solving the mirror constraint equations for LG Calabi–Yau theories in terms of the toric data of the sigma model on M_3, giving directly the right dimension of the mirror geometry without introducing non-dynamical variables. In this way, the mirror LG Calabi–Yau threefolds W_3 can be described as hypersurfaces in four-dimensional weighted projective spaces $\mathbf{WP}^4$, depending on the toric data of M_3. In section 4 we give a toric description of the above-mentioned duality. In particular, we propose a special $\mathbf{Z}_2$ symmetry acting on the toric geometry coordinates leading to G_2 manifolds with $K3$ fibrations. Then we discuss other examples where $\mathbf{Z}_2$ acts trivially on mirror pairs (M_3, W_3). In the last section we give our conclusion.

2. On F-theory duals of M-theory on G_2 manifolds in four dimensions

In this section we study the duality between M-theory on G_2 manifolds and F-theory on elliptically fibred Calabi–Yau fourfolds. To start with, recall that the duality between M-theory and F-theory was studied using different ways. For instance, this can be achieved using the Mayr work based on the local mirror symmetry and special limits in the elliptic compactification of F-theory on Calabi–Yau manifolds [29, 30]. An alternative approach can be adopted using the Horava–Witten compactification on spaces of the form $\frac{S^1}{Z_2} \times Y$, where Y is a Calabi–Yau threefold, giving rise to $N = 1$ supersymmetry in four dimensions [31]. The latter involves a weak coupling limit given by the heterotic superstring compactified on Calabi–Yau threefolds which may have F-theory dual on Calabi–Yau fourfolds. However, in this work we introduce manifolds with G_2 holonomy in the game. In particular, we would like to discuss a new duality between M-theory on G_2 manifolds and F-theory on elliptically fibred Calabi–Yau fourfolds, with $SU(4)$ holonomy group in four dimensions with $N = 1$ supersymmetry. Before going ahead, let us start with the first possible equivalence between M-theory and F-theory. This can appear in nine dimensions with the help of T-duality in type II superstrings. Roughly speaking, this duality can be rewritten, using the above-mentioned theories, as follows

$$\text{M-theory on } S^1 \times S^1(R) = \text{F-theory on } T^2 \times S^1\left(\frac{1}{R}\right) \tag{2.1}$$

where R and $\frac{1}{R}$ are the radii of type II one circle compactifications. In this case, the $Sl(2, \mathbf{Z})$ symmetry in type IIB superstring can also have a geometric realization in terms of the M-theory on elliptic curve T^2. Duality (2.1) can be pushed further for describing the same phenomenon involving spaces that are more complicated than a circle, such as Calabi–Yau spaces in which the T-duality will be replaced by the mirror transformation. Indeed, using mirror symmetry duality in the Calabi–Yau compactifications, equation (2.1) can be extended to

$$\text{M-theory on } S^1 \times M_3 = \text{F-theory on } T^2 \times W_3 \tag{2.2}$$

where (M_3, W_3) are mirror pair manifolds whose Hodge numbers $h^{1,1}$ and $h^{2,1}$ satisfy

$$h^{1,1}(M_3) = h^{2,1}(W_3) \qquad h^{2,1}(M_3) = h^{1,1}(W_3).$$

In this way, the complex (Kahler) structure moduli space of M_3 is identical to the Kahler (complex) structure moduli space of W_3 and the above four-dimensional models are equivalent to type II superstrings compactified on mirror pairs (M_3, W_3). Thus, equation (2.2) describes models with eight supercharges in four dimensions, i.e. $N = 2$ 4D. In what follows, we relate this duality to M-theory on manifolds with G_2 holonomy. However, to do this one has to break the half of the supersymmetry and should look for the expected holonomy groups which are needed on both sides. It turns out that there are some possibilities to realize the first requirement. One way is to use the result of the string compactifications on Calabi–Yau manifolds with Ramond–Ramond fluxes [32]. Another method we are interested in here, is to consider the modding of the above duality by $\mathbf{Z}_2$ symmetry. Note, in passing, that this operation has been used in many cases in string theory compactifications to break the half of the supercharges, particularly, in the case of type IIA propagating on T^4, which is known by an orbifold limit of $K3$ surfaces. The four-dimensional M-theory/F-theory dual pairs with $N = 1$ supersymmetry can be obtained by $\mathbf{Z}_2$ modding of corresponding dual pairs with $N = 2$ given in (2.2). Using this procedure, the resulting space in M-theory compactification is now a quotient space of the following form,

$$X_7 = \frac{S^1 \times M_3}{\mathbf{Z}_2} \tag{2.3}$$

where $\mathbf{Z}_2$ acts on S^1 as a reflection and non-trivially on the Calabi–Yau threefolds M_3. The holonomy group of this geometry is now larger than $SU(3)$ holonomy of M_3, which is the maximal subgroup of the G_2 Lie group. The superstrings propagating on this type of manifold preserve $\frac{1}{8}$ supercharges in three dimensions. Using the decompactification mechanism, M-theory on this geometry has similar features to seven manifolds with G_2 holonomy. This type of manifold has been a subject of intense interest during the last few years, dealing with different problems in superstring theory. In particular, these involve the computation of instanton superpotentials [11], the description of IIA superstring orientifold compactifications giving four-dimensional $N = 1$ models [12] and the study of two-dimensional superconformal field theories [22]. In all these works, the $\mathbf{Z}_2$ acts on the complex homogeneous variables, defining the Calabi–Yau threefolds, by complex conjugation. However, in the present work we will use a new transformation acting on the toric geometry realizations of Calabi–Yau spaces. In section 4 we will show that this procedure leads to $K3$ fibrations in G_2 manifold compactifications. In this way, we will be able to find heterotic superstring dual models using M-theory/heterotic duality in seven dimensions [10, 17, 18].

On the F-theory side, the $N = 1$ dual model may be obtained using the same procedure by taking the following quotient space:

$$W_4 = \frac{T^2 \times W_3}{\mathbf{Z}_2}. \tag{2.4}$$

In this equation, $\mathbf{Z}_2$ acts non-trivially on the mirror geometry W_3 and as a reflection on T^2 as follows,

$$\mathbf{Z}_2 : \quad \mathrm{d}z \to -\mathrm{d}z \tag{2.5}$$

where z is the complex coordinate of the elliptic curve T^2. If this symmetry has some fixed points, they need to be deformed to obtain a smooth manifold. This manifold is then elliptically fibred over $\frac{W_3}{\mathbf{Z}_2}$. As on the M-theory side, the holonomy group of this quotient space is larger than the $SU(3)$ holonomy of W_3. However, the four-dimensional field theories with $N = 1$ supersymmetry obtained from F-theory compactification require that W_4 is an elliptically fibred Calabi–Yau fourfold with $K3$ fibration with $SU(4)$ holonomy. From these physical and mathematical arguments, we can propose, up to some details, the following new duality,

$$\text{M-theory on } X_7(G_2) = \text{F-theory on } W_4(SU(4)) \tag{2.6}$$

where both sides involve mirror pairs of Calabi–Yau threefolds (M_3, W_3). In what follows, we give a comment concerning the relation connecting the Betti numbers of the above quotient manifolds. This can be done only by knowing the results of the numbers of vector multiplets and massless neutral chiral multiplets obtained from both sides. Using the results of [33, 34], we expect the following formula,

$$\begin{aligned} b_2(X_7) &= h^{1,1}(W_4) - h^{1,1}(B_3) - 1 + h^{2,1}(B_3) \\ b_3(X_7) &= h^{1,1}(B_3) - 1 + h^{2,1}(W_4) - h^{2,1}(B_3) + h^{3,1}(W_4) \end{aligned} \tag{2.7}$$

where $B_3 = \frac{W_3}{\mathbf{Z}_2}$. These Betti numbers, in general, depend on the framework of the geometric construction of manifolds and how the $\mathbf{Z}_2$ symmetry acts on the Calabi–Yau threefold. In the present context, the nonzero Betti numbers correspond to the $\mathbf{Z}_2$ invariant forms of the above quotient spaces. In the next section, we will use the toric geometry language, its relation to sigma model and LG mirror geometries to give the expected relations and give some illustrative examples. In this analysis, one has the following Hodge constraint equation,

$$h^{2,1}(M_3) = h^{1,1}(W_3) = 0 \tag{2.8}$$

reducing (2.7) to

$$b_2(X_7) = h^{1,1}(W_4) - 1 + h^{2,1}(B_3)$$
$$b_3(X_7) = h^{2,1}(W_4) - 1 - h^{2,1}(B_3) + h^{3,1}(W_4). \tag{2.9}$$

More precisely, our strategy will be as follows:

(i) Building three-dimensional Calabi–Yau threefolds and their mirrors in terms of hypersurfaces in four-dimensional toric varieties using the toric geometry techniques, their relation to the sigma model and LG theories. In this step, we give a tricky method for obtaining the mirror toric Calabi–Yau threefolds W_3, involved in F-theory compactifications, from the toric data of M_3 manifolds.

(ii) We give a toric description for the duality given in (2.6).

3. Toric geometry for Calabi–Yau threefolds and their mirrors

3.1. Toric geometry of Calabi–Yau threefolds M_3

Complex Calabi–Yau varieties are Ricci flat spaces with a vanishing first Chern class $c_1 = 0$. They are an important ingredient for constructing quasi-realistic superstring models in lower dimensions. As we have seen, they also play a crucial role in the study of the duality between superstring models and other theories; particularly in our proposition given in (2.6). A large class of these manifolds is usually constructed as hypersurfaces in toric varieties and nicely described using toric geometry techniques. Calabi–Yau threefolds are the most important geometries in string theory as well as their mirrors which can be used in the type II duality, the geometric engineering, F-theory-heterotic duality and this proposed work [35–38]. For this reason, we will focus our attention on this special Calabi–Yau geometry. In particular, we develop a tricky way for obtaining the mirror Calabi–Yau manifolds using techniques of toric geometry. In this way, we give the mirror manifolds as hypersurfaces in four-dimensional weighted projective spaces $\mathbf{WP}^4$ of weights $(w_1, w_2, w_3, w_4, w_5)$. As we will see, this construction is based on the solving of the mirror constraint equations involved in the toric geometry in terms of the toric data of the Calabi–Yau threefolds in M-theory compactifications. Before doing this, let us start by describing M_3 in toric geometry framework and its relation to the linear sigma model. This will be useful for our later analysis on geometries involved in the duality between M-theory on G_2 manifolds and F-theory compactifications on elliptically fibred Calabi–Yau fourfolds. We first note that toric geometry is a good tool for describing the essential one needs about the n-dimensional Calabi–Yau manifolds and their mirrors involved in the previous study. Roughly speaking, toric manifolds are complex n-dimensional manifolds with T^n fibration over n real-dimensional base spaces with boundary [39–42]. They exhibit toric actions $U(1)^n$ allowing us to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes Δ_n of the R^n space. In this correspondence, fixed points of the toric actions $U(1)^n$ are associated with the vertices of the polytope Δ_n, the edges are fixed one-dimensional lines of a subgroup $U(1)^{n-1}$ of the toric action $U(1)^n$ and so on. Geometrically, this means that the T^n fibres can degenerate over the boundary of the base. Note that in the case where the base space is compact, the resulting toric manifold will also be compact. The beauty of the toric representation is that it permits us to learn the essential about the geometric features of toric manifolds by simply knowing the toric data of the corresponding polytope Δ_n, involving the toric vertices and the Mori vector weights.

A simple example of a toric manifold is $\mathbf{C}^n$, which can be parametrized by $z_i = |z_i|\,\mathrm{e}^{\mathrm{i}\theta_i}$, $i = 1, \ldots, n$ and endowed with Kahler form given by

$$J = \mathrm{i}\mathrm{d}\bar{z}_i \wedge \mathrm{d}z_i = \mathrm{d}(|z_i|^2) \wedge \mathrm{d}\theta_i. \tag{3.1}$$

This manifold admits $U(1)^n$ toric actions

$$z_i \longrightarrow z_i\, e^{i\theta_i} \tag{3.2}$$

with fixed locus at $z_i = 0$. The geometry of $\mathbf{C}^n$ can be represented by a T^n fibration over an n-dimensional real space parametrized by $|z_i|^2$. The boundary of the base is given by the union of the hyperplanes $|z_i|^2 = 0$. We have given a very simple example of toric manifolds. However, toric geometry is also very useful for building (local) Calabi–Yau manifolds, providing a way for superstring theory to interesting physics in lower dimensions, in particular four dimensions. An interesting example is the asymptotically local Euclidean (ALE) space with ADE singularities. These are local complex two-dimensional toric varieties with an $SU(2)$ holonomy. However, for later use, we will restrict ourselves to local three complex Calabi–Yau manifolds M_3 and their mirrors denoted by W_3 with an $SU(3)$ holonomy. The toric Calabi–Yau M_3, involved in M-theory compactification, can be represented by the following algebraic equation,

$$\sum_{i=1}^{r+3} Q_i^a |z_i|^2 = R^a \tag{3.3}$$

together with the local Calabi–Yau condition

$$\sum_{i=1}^{r+3} Q_i^a = 0 \qquad a = 1, \ldots, r. \tag{3.4}$$

In equations (3.3) and (3.4), Q_i^a are integers defining the weights of the toric actions of the complex manifold in which the M_3 is embedded. Actually these equations, up to some details on Q_i^a, generalize that of the weighted projective space with weights Q_i corresponding to $r = 1$. Each parameter R^a is a Kahler deformation of the Calabi–Yau manifolds. The above geometry can be encoded in a toric diagram $\Delta(M_3)$ having $r + 3$ vertices v_i generating a finite-dimensional sublattice of the $\mathbf{Z}^5$ lattice and satisfying the following r relations given by

$$\sum_{i=1}^{r+3} Q_i^a v_i = 0 \qquad a = 1, \ldots, r \tag{3.5}$$

with the local Calabi–Yau condition (3.4).

Equations (3.3) and (3.5) can be related to the so-called D-term potential of the two-dimensional $N = 2$ sigma model, for putting the discussion on a physical framework [43]. Indeed associating the previous variables z_i, or v_i vertices in toric geometry language, (ϕ_i) matter fields and interpreting the Q_i^a integers as the quantum charges the (ϕ_i) under a $U(1)^r$ symmetry, then the toric Calabi–Yau M_3 is now the moduli space of the 2D $N = 2$ supersymmetric linear sigma model. The Q_i^a obey naturally the neutrality condition, being equivalent to $c_1(M_3) = 0$, which means that the theory flows in the infrared to a non-trivial superconformal model [43, 44]. In this way, equation (3.3) can be identified with the D-flatness conditions, namely

$$\sum_{i=1}^{r+3} Q_i^a |\phi_i|^2 = R^a. \tag{3.6}$$

In these equations, R^a are FT terms which can be complexified by the θ angles as follows

$$t^a = R^a + \theta^a \tag{3.7}$$

where θ^a have a role similar to the B field in the string theory compactification on Calabi–Yau manifolds. The number of independent FI parameters, or equivalently the number of $U(1)$ factors, equals $h^{1,1}(M_3)$.

3.2. Solving the mirror constraint equations for W_3

Toric geometry has been adopted to discuss mirror symmetry also. The latter exchanges the Kahler structure parameters with the complex structure parameters. In general, given a toric realization of the manifold M_3, one can build its mirror manifold W_3. This will also be a toric variety which is obtained from M_3 with the help of mirror symmetry. In this study, we will use the result of mirror symmetry in the sigma model where the mirror Calabi–Yau W_3 will be an LG Calabi–Yau superpotential, depending on the number of chiral multiples and gauge fields of the dual sigma model on M_3. A tricky way to write down the equation of the LG mirror Calabi–Yau superpotential is to use dual chiral fields Y_i related to sigma model fields such that [45–48]

$$\mathrm{Re}\, Y_i = |\phi_i|^2 \qquad i = 1, \ldots, k \tag{3.8}$$

and define the new variables y_i as follows: $y_i = \mathrm{e}^{-Y_i}$. The W_3 LG mirror Calabi–Yau superpotential takes the form

$$\sum_{i=1}^{r+3} y_i = 0 \tag{3.9}$$

where the fields y_i must satisfy, up to absorbing the complex Kahler parameters t^a, the following constraint equations

$$\prod_{i=1}^{r+3} y_i^{Q_i^a} = 1 \qquad a = 1, \ldots, r. \tag{3.10}$$

In toric geometry language, this means that the relation between the toric vertices of M_3 maps to relations given by (3.9) and (3.10). To find an explicit algebraic equation for the local mirror geometry, one has to solve the constraint equation (3.10). It turns out that there are many ways to solve these constraint equations. Here we present a tricky way, inspired by the Batyrev papers [49, 50] and [37], using the toric geometry representation of the sigma model of M_3. This method can proceed in some steps. First, we note that not all the y_i are independent variables, only four of them are. The latter can be thought of as local coordinates of the weighted projective space $\mathbf{WP}^4(w_1, w_2, w_3, w_4, w_5)$ which is parametrized by the following five homogeneous variables:

$$x_\ell = \lambda^{w_\ell} x_\ell \qquad \lambda \in C^* \quad \ell = 1, \ldots, 5. \tag{3.11}$$

For instance, the four local variables can be obtained from the homogeneous ones using some coordinate patch. If $x_5 = 1$, the other x_ℓ variables behave as four independent gauge invariants under C^* action of $\mathbf{WP}^4(w_1, w_2, w_3, w_4, w_5)$. The second step in our program is to find relations between the y_i and the x_i variables. A nice way of obtaining this is based on using the toric data of the M-theory Calabi–Yau geometry for solving the mirror constraint equations (3.10). In this method, the mirror geometry W_3 will be defined as a D degree homogeneous hypersurfaces in $\mathbf{WP}^4$ with the following form

$$p_D(x_1, x_2, x_3, x_4, x_5) = 0 \tag{3.12}$$

satisfying

$$p_D(\lambda^{w_\ell} x_\ell) = \lambda^D p_D(x_\ell). \tag{3.13}$$

To write down the explicit formula of this equation, one has to solve the mirror constraint equation (3.10) in terms of the WP^4 toric data. For that purpose, we consider a solution of the dual toric manifold M_3 of the form

$$\sum_{i=1}^{r+3} Q_i^a n_i^\ell = 0 \qquad a = 1, \ldots, r \quad \ell = 1, \ldots, 5 \tag{3.14}$$

where n_i^ℓ are integers specified later on. In patch coordinates $x_5 = 1$, one can parametrize the y_i gauge invariants in terms of the x_i as follows,

$$y_i = x_1^{(n_i^1-1)} x_2^{(n_i^2-1)} x_3^{(n_i^3-1)} x_4^{(n_i^4-1)} x_5^{(n_i^5-1)} = \prod_{\ell=1}^{5} x_\ell^{(n_i^\ell-1)} \tag{3.15}$$

where the deformation given by $y_0 = 1$ corresponds to $\left(n_0^\ell - 1\right) = 0$. Using the toric geometry data of M_3, (3.10) is trivially satisfied by (3.15). Another thing we need in this analysis is that the y_i variables should be thought of as gauge invariants under the $\mathbf{WP}^4(w_1, w_2, w_3, w_4, w_5)$ projective action given by (3.11). Indeed under this transformation, the monomials y_i transform as

$$y_i = \Pi_{\ell=1}^5 x_\ell^{(n_i^\ell-1)} \to y_i{}' = y_i \lambda^{\Sigma_\ell (w_\ell (n_i^\ell - 1))} \tag{3.16}$$

and their invariance is constrained by

$$\sum_{\ell=1}^{5} w_\ell = D \tag{3.17}$$

$$\sum_{\ell=1}^{5} w_\ell n_i^\ell = D. \tag{3.18}$$

Equation (3.17) is a strong constraint which will be necessary for satisfying the Calabi–Yau condition in the mirror geometry; while equation (3.18) shows that the n_i^ℓ integers involved in (3.14) and (3.15) can be solved in terms of the partitions d_i^ℓ of the degree D of the homogeneous polynomial $p_D(x_1, \ldots, x_5)$. Since $\sum_{\ell=1}^5 d_i^\ell = D$, one can see that $n_i^\ell = \frac{d_i^\ell}{w_\ell}$; and take, for $i = \ell$, the following property:

$$n_i^\ell = \frac{D}{w_\ell} \qquad i = 1, \ldots, \ell. \tag{3.19}$$

In this way the v_i vertices can be chosen as follows,

$$v_i^\ell = n_i^\ell - e_0^\ell = \frac{d_i^\ell}{w_\ell} - e_0^\ell \tag{3.20}$$

where $e_0^\ell = (1, 1, 1, 1, 1)$. This shifting will not influence the toric realization (3.15) due to the Calabi–Yau condition (3.4). In this way the first six vertices and the corresponding monomials can be thought of as follows:

$$\begin{aligned}
v_0 &= (0, 0, 0, 0, 0) \to \prod_{\ell=1}^{5} (x_\ell) \\
v_1 &= \left(\frac{D}{w_1} - 1, -1, -1, -1, -1\right) \to x_1^{\frac{D}{w_1}} \\
v_2 &= \left(-1, \frac{D}{w_2} - 1, -1, -1, -1\right) \to x_2^{\frac{D}{w_2}} \\
v_3 &= \left(-1, -1, \frac{D}{w_3} - 1, -1, -1\right) \to x_3^{\frac{D}{w_3}} \\
v_4 &= \left(-1, -1, -1, \frac{D}{w_4} - 1, -1\right) \to x_4^{\frac{D}{w_4}} \\
v_5 &= \left(-1, -1, -1, -1, \frac{D}{w_5} - 1\right) \to x_5^{\frac{D}{w_5}}.
\end{aligned} \tag{3.21}$$

Using all the toric vertices in the M-theory geometry, the corresponding mirror polynomial should involve in the F-theory context the following form,

$$\sum_{\ell=1}^{5} x_\ell^{\frac{D}{w_\ell}} + a_0 \prod_{\ell=1}^{5} (x_\ell) + \sum_{i=7}^{r+3} a_i \prod_{\ell=1}^{5} x_\ell^{n_i^\ell} = 0 \tag{3.22}$$

where the a_i are complex moduli of the LG Calabi–Yau mirror superpotentials. For later use, we take $a_i = 0$ and so the above geometry reduces to

$$\sum_{\ell=1}^{5} x_\ell^{\frac{D}{w_\ell}} + a_0 \prod_{\ell=1}^{5} (x_\ell) = 0. \tag{3.23}$$

Actually, this geometry extends the quintic hypersurfaces in the ordinary $\mathbf{P}^4$ projective space.

4. $\mathbf{Z}_2$ symmetry in toric geometry framework

4.1. $\mathbf{Z}_2$ realization and $K3$ fibration

Here we would like to discuss the $\mathbf{Z}_2$ realization involved in the duality (2.6) using toric geometry tools. The latter, in the G_2 holonomy sense, will act on M_3, on the circle as the inversion, on the T^2 and on W_3 in the F-theory compactification. Due to the richness of possibilities of the $\mathbf{Z}_2$ action in the Calabi–Yau manifold, we will focus our attention below on giving a new $\mathbf{Z}_2$ transformation acting on the toric geometry variables. For this purpose, we first note that any local Calabi–Yau M_3, described by the toric linear sigma model on (3.3) is, up to some details, isomorphic to C^{r+3}/C^{*r}, or equivalently

$$z_i \equiv \lambda^{Q_i^a} z_i \qquad \sum_{i=1}^{r+3} Q_i^a = 0 \qquad a = 1, \ldots, r. \tag{4.1}$$

The latter has a T^3 fibration obtained by dividing T^{r+3} by $U(1)^r$ action generated by a simultaneous phase rotation of the coordinates

$$z_i \to \mathrm{e}^{\mathrm{i}Q_i^a \vartheta^a} z_i \qquad a = 1, \ldots, r \tag{4.2}$$

where ϑ^a are the generators of the $U(1)$ factors. In the Calabi–Yau geometry, plus the reflection on the circle, we will consider a new $\mathbf{Z}_2$ symmetry acting on the toric geometry angular coordinates. The latter leads to G_2 manifolds with $K3$ fibration on the M-theory side. To see this feature, let us first consider the simple case corresponding to $\mathbf{C}^3$, that is $r = 0$; then we extend this feature to any three-dimensional toric varieties. Indeed, $\mathbf{C}^3$ has $U(1)^3$ toric actions giving T^3 fibration in a toric geometry realization of $\mathbf{C}^3$. Besides these toric actions, consider now a $\mathbf{Z}_2$ symmetry acting as follows:

$$\theta \longrightarrow -\theta_i \qquad i = 1, 2, 3. \tag{4.3}$$

In this way, the toric actions now become

$$z_i \longrightarrow z_i\, \mathrm{e}^{\mathrm{i}\theta_i} \qquad \theta_i \longrightarrow -\theta_i \qquad i = 1, 2, 3 \tag{4.4}$$

where z_i are the variables appearing in (3.3). This transformation is quite different from that given in the literature [11, 12, 22], because it acts on the angular variables of complex toric varieties. Note, in passing, that one can consider the following $\mathbf{Z}_2$,

$$\theta_i \longrightarrow \theta_i + \pi \qquad i = 1, 2, 3;$$

however, this transformation is not interesting from physical arguments. Indeed, first this action has no fixed points because the fixed loci are naturally identified with brane configurations [40]. Second, it does not leave the following constraint equation,

$$\theta_1 + \theta_2 + \theta_3 = 0 \tag{4.5}$$

involved in the determination of special Lagrangian manifolds in $\mathbf{C}^3$ [47].

Now we return to equation (4.4). The latter gives naturally $\frac{T^3}{\mathbf{Z}_2}$ as a fibre space in the toric geometry realization of $\mathbf{C}^3$, unlike the before where we have only a T^3 fibration. This feature can be extended to any three-dimensional toric complex manifolds, in particular local Calabi–Yau threefolds M_3. In this way, one has the following $\mathbf{Z}_2$ symmetry acting, up to the gauge transformation (4.2), on the Calabi–Yau M_3 variables as follows:

$$z_i \longrightarrow z_i\,\mathrm{e}^{\mathrm{i}\theta_i} \qquad \theta_i \longrightarrow -\theta_i \qquad i = 1, \ldots, r+3. \tag{4.6}$$

Geometrically, this transformation gives a local Calabi–Yau threefold with $\frac{T^3}{\mathbf{Z}_2}$ fibration. In this case, in addition to the toric geometry action fixed points we now have extra ones coming from the orbifold toric fibration. These fixed loci may be identified with brane configurations using the interplay between toric geometry and type II brane configurations [40]. For the moment we ignore the brane description and return to the geometric interpretation of (4.6). Indeed, it is easy to see that, together with the action on the circle, these new toric actions lead to G_2 manifolds with $\frac{T^4}{\mathbf{Z}_2}$ fibration, being the orbifold limit of the $K3$ surfaces, in the M-theory compactifications. The geometry given by (2.3) can now be viewed as a G_2 manifold with $K3$ fibration. In this way, we are able to find heterotic superstring dual models which could be used to support our proposed duality (2.6). Indeed, the moduli space of smooth compactifications can be obtained from that of $K3$ followed by an extra compactification on a three-dimensional space Q_3, down to four dimensions[1]. This can be related directly to the heterotic superstring by fibring the M-theory/heterotic duality in seven dimensions on the same base Q_3. Locally, the moduli space of this compactification should have the following form,

$$\mathcal{M}(K3) \times \mathcal{M}(Q_3) \tag{4.7}$$

where $\mathcal{M}(K3)$ is the moduli space of the M-theory on $K3$ in seven dimensions,

$$\mathcal{M}(K3) = \mathbf{R}^+ \times \frac{SO(3, 19)}{SO(3) \times SO(19)}$$

which is exactly the moduli space of heterotic strings on T^3. $\mathcal{M}(Q_3)$ describes the physical moduli coming after the extra compactification on X_3. This compactification describes the strong limit of heterotic superstrings on Calabi–Yau threefolds Z, being a T^3 fibration on Q_3. More recently, it was shown that M-theory on a G_2 manifold with $K3$ fibration can give much more interesting physics than other superstring derived models; in particular, it leads to a theory similar to a four-dimensional grand unified model [24]. Alternatively, the $\frac{T^4}{\mathbf{Z}_2}$ fibre space is locally isomorphic to $\frac{\mathbf{C}^2}{\mathbf{Z}_2}$ known by A_1 singularity, which can be determined algebraically in terms of the $\mathbf{Z}_2$ invariant coordinates on $\mathbf{C}^2$ as follows:

$$z^2 = xy. \tag{4.8}$$

M-theory on this local geometry singularity corresponds to two units of D6 branes. For the general case where we have A_n singularity, this geometry is equivalent to $n + 1$ D6 branes of type IIA superstring. In this way, the above compactification may have a four-dimensional interpretation in terms of type IIA D6 branes.

[1] The G_2 holonomy and the Calabi–Yau condition of $K3$ require that $b_1(Q_3) = 0$.

Before going to F-theory, we discuss the Rahm cohomology of the M-theory quotient space. The Calabi–Yau condition $b_1(M_3) = 0$ and the G_2 holonomy condition require that $b_1(X_7) = 0$. Using equations (4.6), the Kahler form now is odd under the above $\mathbf{Z}_2$ symmetry. Since there are no invariant 2-forms, we have the following constraint for the quotient space

$$b_2 = 0. \tag{4.9}$$

However, there are some invariant 3-forms; one type is given by

$$\phi = J \wedge \mathrm{d}x \tag{4.10}$$

where J is the Kahler form on M_3 and x is a real coordinate parametrizing the circle. The number of these forms is given by $h^{1,1}(M_3)$ being the dimension of the complexified Kahler moduli space of M_3.

Now we go on to the F-theory to give the corresponding geometries. Instead of being general, we will consider a concrete example describing the mirror quintic hypersurfaces obtained by taking

$$w_\ell = 1 \quad \forall \ell. \tag{4.11}$$

In this way, the general mirror geometry (3.23) reduces to

$$\sum_{\ell=1}^{5} x_\ell^5 + a_0 \prod_{\ell=1}^{5} (x_\ell) = 0. \tag{4.12}$$

This equation has any direct toric description of the $\mathbf{P}^4$ in which it is embedded. However, a toric realization may be recovered if we consider the limit $a_0 \to \infty$ in the mirror description. In this limit, the defining equation of the mirror geometry becomes approximately, up to scaling out the a_0,

$$x_1 \ldots x_5 = 0. \tag{4.13}$$

This equation can be solved by taking one or more $x_i = 0$. In toric geometry language, this solution describes the union of the boundary faces of a 4-simplex defining the polytope of the $\mathbf{P}^4$ projective space [40]. In this case, the F-theory mirror quintic is a T^3 fibration over three-dimensional real space defined by the boundary faces of a 4-simplex of $\mathbf{P}^4$. It consists of the intersection of five $\mathbf{P}^3$ along ten $\mathbf{P}^2$. This geometry has now a $U(1)^3$ toric action which can be deduced from those of $\mathbf{P}^4$ and can be thought of as follows:

$$\begin{aligned} x_i &\to \mathrm{e}^{\mathrm{i}\theta_i} x_i \qquad & i &= 1, 2, 3 \\ x_i &\to x_i \qquad & i &= 4, 5. \end{aligned} \tag{4.14}$$

In the F-theory geometry, $\mathbf{Z}_2$ symmetry will act on the mirror Calabi–Yau threefolds as follows

$$\begin{aligned} x_i &\longrightarrow x_i\, \mathrm{e}^{\mathrm{i}\theta_i} \qquad i = 1, 2, 3 \\ \theta_i &\longrightarrow -\theta_i. \end{aligned} \tag{4.15}$$

Here we repeat the same analysis as for M-theory. In this case, the F-theory geometry can also have $\frac{T^4}{\mathbf{Z}_2}$ fibration over a four-dimensional base space. However, a naive way to obtain an elliptically $K3$ fibration, being the relevant geometry in the F-theory compactification for obtaining $N = 1$ in four dimensions, is to suppose that the $\mathbf{Z}_2$ symmetry acts trivially on the torus of W_4 Calabi–Yau fourfolds. In this way, the $\frac{T^4}{\mathbf{Z}_2}$ fibration reduces to $T^2 \times \frac{T^2}{\mathbf{Z}_2}$ which is an elliptic model in the context of F-theory compactifications. This compactification should be interpreted in terms of type IIB on $\frac{T^2}{\mathbf{Z}_2}$. By this limit, one can see that the orbifold (2.4) gives an elliptic $K3$ fibration in F-theory compactifications. In the context of M-theory, this geometry can be obtained using the following factorization in the Narain lattice,

$$\Gamma^{19,3} = \Gamma^{18,2} + \Gamma^{1,1} \tag{4.16}$$

having a nice interpretation in terms of the action of $\mathbf{Z}_2$ symmetry on the moduli space of $K3$ surfaces [51].

4.2. More on the $\mathbf{Z}_2$ action

We would like to make comments regarding a particular realization of $\mathbf{Z}_2$ symmetry when it acts trivially on the Calabi–Yau threefolds. In this way, the geometries (2.3) and (2.4) reduce respectively to

$$X_7 = \frac{S^1}{\mathbf{Z}_2} \times M_3 \tag{4.17}$$

$$W_4 = \frac{T^2}{\mathbf{Z}_2} \times W_3. \tag{4.18}$$

In M-theory, this compactification may be thought of as the Hořava–Witten compactification on spaces of the form $\frac{S^1}{\mathbf{Z}_2} \times Y$, where $Y = M_3$ is a Calabi–Yau threefold [52]. M-theory on this type of manifold gives rise to $N = 1$ supersymmetry in four dimensions, having a weak coupling limit given by the heterotic superstring compactified on M_3. Using the toric description of M_3 where one has a T^3 fibration over three-dimensional base space, the compactification (4.12) may be related to M-theory on G_2 manifolds with $K3$ fibration. This may be checked by the seven-dimensional duality between M-theory on $K3$ and the heterotic superstring on T^3. At the end of this section, we discuss the F-theory duals of this kind of compactification. Instead of being general, we will consider concrete examples corresponding to M_3, described by the $N = 2$ linear sigma model on the canonical line bundle over two complex-dimensional toric varieties. In F-theory, the mirror maps of these geometries are given by non-compact Calabi–Yau threefold LG superpotentials with an equation of the form

$$W_3 : \quad f(x_1, x_2) = uv \tag{4.19}$$

where x_1, x_2 are C^* coordinates and u, v are C^2 coordinates. For illustrative applications, let us give two examples.

(i) $\mathbf{P}^2$ *projective space*. The first example is the sigma model on the canonical line bundle over $\mathbf{P}^2$. In this way, the Calabi–Yau geometry on the M-theory side is described by a $U(1)$ linear sigma model with four matter fields ϕ_i with the following vector charge,

$$Q_i = (1, 1, 1, -3), \tag{4.20}$$

satisfying the Calabi–Yau condition (3.4). After solving the mirror constraint equations (3.10), the corresponding W_3 LG Calabi–Yau superpotential in F-theory compactification is given by the following equation:

$$W_3 : \quad f(x_1, x_2) = 1 + x_1 + x_2 + \frac{\mathrm{e}^{-t}}{x_1 x_2} = uv. \tag{4.21}$$

(ii) *Hirzebruch surfaces* $\mathbf{F}_n$. As a second example, we consider a local model given by the canonical line bundle over the Hirzebruch surfaces $\mathbf{F}_n$. Recall by the way that the $\mathbf{F}_n$ surfaces are two-dimensional toric manifolds, defined by a non-trivial fibration of a $\mathbf{P}^1$ fibre on a $\mathbf{P}^1$ base. The latter are realized as the vacuum manifold of the $U(1) \times U(1)$ gauge theory with four chiral fields with charges

$$Q_i^{(1)} = (1, 1, 0, -n) \qquad Q_i^{(2)} = (0, 0, 1, 1). \tag{4.22}$$

The canonical line bundle over these surfaces is a local Calabi–Yau threefold described by a $U(1) \times U(1)$ linear sigma model with five matter fields ϕ_i with two vector charges

$$Q_i^{(1)} = (1, 1, 0, -n, n-2) \qquad Q_i^{(2)} = (0, 0, 1, 1, -2). \tag{4.23}$$

These satisfy naturally the Calabi–Yau condition (3.4). After solving (3.10), the defining equation for the LG mirror superpotential becomes

$$W_3 : f_n(x_1, x_2) = uv \tag{4.24}$$

where $f_n(x_1, x_2) = 1 + x_1 + \frac{e^{-t_1} x_2^n}{x_1} + x_2 + \frac{e^{-t_2}}{x_4}$ and where t_i are complex parameters.

On the F-theory side, equations (4.21) and (4.24) describe elliptic fibration solutions of (4.19), where the elliptic curve fibre is given by

$$f(x_1, x_2) = 0. \tag{4.25}$$

By introducing an extra variable, this elliptic curve can have a homogeneous representation described by the cubic polynomial in $\mathbf{P}^2$, with the general form as follows,

$$\sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0 \tag{4.26}$$

which can be related, up to some limits in the complex structures, to the following Weierstrass form:

$$y^2 z = x^3 + axz^2 + bz^3. \tag{4.27}$$

This form plays an important role in the construction of elliptic Calabi–Yau manifolds involved in F-theroy compactifications [25, 26, 48].

The elliptic $K3$ fibration of (4.8) may be obtained by using only the orbifold of the torus in F-theory. A tricky way to see this is as follows. First, we consider the elliptic curve T^2 as a fibre circle S_f^1 of radius R_f, fibred on an S_b^1 base circle of radius R_b. After that we take a $\mathbf{Z}_2$ symmetry acting only on the S_b^1 and leaving the S_f^1 fibre invariant. In this way, the orbifold $\frac{T^2}{\mathbf{Z}_2}$ can be seen as an S_f^1 bundle over a line segment, with two fixed points—zero and πR_b. This space, up to some details, has similar features to the toric realization of $\mathbf{P}^1$. To see this, assume that the radius of the S_f^1 fibre varies on the base space as follows,

$$R_f \sim \sin \frac{x}{R_b} \tag{4.28}$$

where x is the coordinate on the interval which runs from zero to πR_b. In this case, the S_f^1 can shrink at the two end points of the line segment. With this assumption, the resulting geometry is identified with the toric geometry realization of the $\mathbf{P}^1$ [39, 40]. By this limit in the $\mathbf{Z}_2$ orbifold, the F-theory geometry given by (4.18) can be viewed as a $\mathbf{P}^1$ fibration over W_3. Since W_3 is an elliptic model, this fibration, in the presence of $\mathbf{P}^1$, can give an elliptic $K3$ fibration which is the relevant geometry for getting $N = 1$ models in four dimensions from F-theory compactifications.

5. Discussions and conclusion

In this paper, we have studied, geometrically, the $N = 1$ four-dimensional duality between M-theory on G_2 manifolds and F-theory on elliptically fibred Calabi–Yau fourfolds with $SU(4)$ holonomy. In this study, we have considered M-theory on spaces of the form $\frac{S^1 \times M_3}{\mathbf{Z}_2}$, where M_3 is a Calabi–Yau threefold described physically by the $N = 2$ sigma model in two dimensions. In particular, using mirror symmetry, we have discussed the possible duality between M-theory on such spaces and F-theory on $\frac{T^2 \times W_3}{\mathbf{Z}_2}$, where (M_3, W_3) are mirror pairs in type II superstring compactifications on Calabi–Yau threefolds. In this work, our results are summarized as follows:

(1) First, we have developed a way of obtaining LG Calabi–Yau threefolds W_3 mirror to the sigma model on toric Calabi–Yau M_3. This method is based on solving the mirror constraint equations for LG theories in terms of the toric data of the sigma model on M_3. Actually, this way gives directly the right dimensions of the mirror geometry. In particular, we have shown that the mirror LG Calabi–Yau threefolds W_3 can be described as hypersurfaces in four-dimensional weighted projective spaces $\mathbf{WP}^4$, depending on the toric data of M_3. In this way, the $\mathbf{WP}^4$ may be determined in terms of the toric geometry data of M_3.

(2) Using these results, we have shown, at least, that there exist two classes of F-theory duals of M-theory on $\frac{S^1 \times M_3}{\mathbf{Z}_2}$. These classes depend on the possible realizations of the $\mathbf{Z}_2$ actions on M_3. In the case where $\mathbf{Z}_2$ acts non-trivially on M_3, we have given a toric description of the above-mentioned duality. In particular, we have proposed a special $\mathbf{Z}_2$ symmetry acting on the toric geometry angular variables. This way gives seven manifolds with $K3$ fibrations of G_2 holonomy. For the case where $\mathbf{Z}_2$ acts trivially on M_3, we have given some illustrative examples of the F-theory duals of M-theory. Finally, since every supersymmetric intersecting brane dynamics is expected to lift to M-theory on local G_2 manifolds [53], it would be interesting to explore this physics using intersecting D6-7 branes in Calabi–Yau manifolds. Moreover, the $N = 1$ model studied in this work can be related to heterotic superstrings on Calabi–Yau threefolds with the elliptic fibration over del Pezzo surfaces $\mathrm{d}P_k$, the $\mathbf{P}^2$ blown up at k points with $k \leqslant 9$. In a special limit, these surfaces may be specified by the elliptic fibration $\frac{W_3}{\mathbf{Z}_2} \to \mathrm{d}P_k$ with $\mathbf{P}^1$ fibres. It should also be interesting to explore physical realizations, via branes, of this fibration. Progress in this direction will be reported elsewhere.

Acknowledgments

I would like to thank many people. I would like to thank Instituto de Fisica Teorica, Universidad Autonoma de Madrid for kind hospitality during the preparation of this work. I am very grateful to C Gómez for discussions, encouragement and scientific help. I would like to thank A M Urenga for many valuable discussions and comments on this work during my stay at IFT-UAM. I am very grateful to E H Saidi for discussions, encouragement and scientific help. I would like to thank the organizers of the Introductory School on String Theory (2002), the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for kind hospitality. I would like to thank K S Narain, M Blau and T Sarkar for discussions at ICTP. I am very grateful to V Bouchard for the paper [14]. I would like to thank J McKay and A Sebbar for discussions, encouragement and scientific help. This work is partially supported by SARS, programme de soutien ă la recherche scientifique, Université Mohammed V-Agdal, Rabat. I would like to thank my family for help.

References

[1] Morrison D R and Plesser M R 1995 *Nucl. Phys.* B **440** 279–354 (*Preprint* hep-th/9412236)
[2] Greene B R, Morrison D R and Plesser M R 1995 *Commun. Math. Phys.* **173** 559–98 (*Preprint* hep-th/9402119)
[3] Ferrara S, Harvey J, Strominger A and Vafa C 1995 Second quantized mirror symmetry *Phys. Lett.* B **361** 59–65 (*Preprint* hep-th/9505162)

[4] Vafa C 1997 Lectures on strings and dualities *Preprint* hep-th/9702201
[5] Kachru S and Vafa C 1995 Exact results for $N = 2$ compactifications of heterotic strings *Nucl. Phys.* B **450** 69–89 (*Preprint* hep-th/9505105)
[6] Witten E 1995 String theory dynamics in various dimensions *Nucl. Phys.* B **443** 85 (*Preprint* hep-th/9503124)
[7] Townsend P K 1995 The eleven-dimensional supermembrane revisited *Phys. Lett.* B **350** 184 (*Preprint* hep-th/9501068)
[8] Gómez C 2001 *Workshop on Noncommutative Geometry, Superstrings and Particle Physics (Rabat, Morocco, May 11–12)*
[9] Townsend P K and Papadopoulos G 1995 Compactification of $D = 11$ supergravity on spaces of exceptional holonomy *Phys. Lett.* B **357** 472 (*Preprint* hep-th/9506150)
[10] Acharya B 2000 On realizing $N = 1$ super Yang–Mills in M-theory *Preprint* hep-th/0011089
[11] Harvey J A and Moore G 1999 Superpotentials and membrane instantons *Preprint* hep-th/9907026
[12] Kachru S and McGreevy J 2001 M-theory on manifolds of G_2 holonomy and type IIA orientifolds *J. High Energy Phys.* JHEP06(2001)027 (*Preprint* hep-th/0103223)
[13] Atiyah M F and Witten E 2001 M-theory dynamics on a manifold of G_2 holonomy *Preprint* hep-th/0107177
[14] Cvetic M, Shiu G and Uranga A M 2001 Chiral type II orientifold constructions as M theory on G_2 holonomy spaces *Preprint* hep-th/0111179
[15] Uranga A M 2002 Localized instabilities at conifolds *Preprint* hep-th/0204079
[16] Joyce D 2000 *Compact Manifolds of Special Holonomy* (Oxford: Oxford University Press)
[17] Witten E 2001 Anomaly cancellation on manifolds of G_2 holonomy *Preprint* hep-th/0108165
[18] Acharya B and Witten E 2001 Chiral fermions from manifolds of G_2 holonomy *Preprint* hep-th/0109152
[19] Curio G 2002 Superpotentials for M-theory on a G_2 holonomy manifold and triality symmetry *Preprint* hep-th/0212211
[20] Friedmann T and Witten E 2002 Unification scale, proton decay, and manifolds of G_2 holonomy *Preprint* hep-th/0211269
[21] Belhaj A 2002 Manifolds of G_2 holonomy from $N = 4$ sigma model *J. Phys. A: Math. Gen.* **35** 8903–12 (*Preprint* hep-th/0201155)
[22] Blumenhagen R and Braun V 2001 Superconformal field theories for compact G_2 manifolds *J. High Energy Phys.* JHEP12(2001)006 (*Preprint* hep-th/0110232)
Roiban R, Romelsberger C and Walcher J 2002 Discrete torsion in singular G_2-manifolds and real LG *Preprint* hep-th/0203272
[23] Gukov S, Yau S T and Zaslow E 2002 Duality and fibrations on G_2 manifolds *Preprint* hep-th/0203217
[24] Witten E 1986 New issues in manifolds of $SU(3)$ holonomy *Nucl. Phys.* B **268** 79–112
Sen A 1987 Heterotic string theory and Calabi–Yau manifolds in the Green–Schwarz formalism *Nucl. Phys.* B **355** 423
[25] Vafa C 1996 *Nucl. Phys.* B **469** 403
[26] Vafa C and Morrison D 1996 *Nucl. Phys.* B **476** 437
[27] Berglund P and Mayr P 1998 Heterotic string/F-theory duality from mirror symmetry *Preprint* hep-th/9811217
Berglund P and Mayr P 1999 Stability of vector bundles from F-theory *J. High Energy Phys.* JHEP12(1999)009 (*Preprint* hep-th/9904114)
[28] Lerche W 1999 On the heterotic/F-theory duality in eight dimensions *Proc. Cargese 1999* (*Preprint* hep-th/9910207)
Andreas B and Curio G 2000 Horizontal and vertical five-branes in heterotic/F-theory duality *J. High Energy Phys.* JHEP01(2000)013 (*Preprint* hep-th/9912025)
[29] Mayr P 1999 Non-perturbative $N = 1$ string vacua *The Spring Workshop on Superstrings and Related Matters (March)* (Trieste: ICTP)
[30] Mayr P 1999 $N = 1$ heterotic string vacua from mirror symmetry *Preprint* hep-th /9904115
[31] Marquart M and Waldram D 2002 F-theory duals of M-theory on $\frac{S^1}{Z_2} \times T^4/Z_N$ *Preprint* hep-th/0204228
[32] Taylor T R and Vafa C 2000 RR flux on Calabi–Yau and partial supersymmetry breaking *Phys. Lett.* B **474** 130–7 (*Preprint* hep-th/9912152)
Gukov S and Haack M 2002 IIA string theory on Calabi–Yau fourfolds with background fluxes *Preprint* hep-th/0203267
[33] Andreas B, Curio G and Lust D 1997 $N = 1$ dual string pairs and their massless spectra *Nucl. Phys.* B **507** 175–96 (*Preprint* hep-th/9705174)
[34] Curio G and Lust D 1997 A class of $N = 1$ dual string pairs and its modular superpotential *Int. J. Mod. Phys.* A **12** 5847–66 (*Preprint* hep-th/9703007)
[35] Katz S, Mayr P and Vafa C 1998 Mirror symmetry and exact solution of 4d $N = 2$ gauge theories I *Adv. Theor. Math. Phys.* **1** 53
[36] Mayr P 1999 Geometric construction of $N = 2$ of gauge theories *Fortschr. Phys.* **47** 39–63

Mayr P 1999 $N = 2$ of gauge theories *Spring School on Superstring Theories and Related Matters (Italy)* (Trieste: ICTP)

[37] Belhaj A, Fallah A E and Saidi E H 2000 On the non-simply mirror geometries in type II strings *Class. Quantum Grav.* **17** 515–32

[38] Belhaj A and Saidi E H 2000 Toric geometry, enhanced non-simply laced gauge symmetries in superstrings and F-theory compactifications *Preprint* hep-th/0012131

[39] Fulton W 1993 *Introduction to Toric Varieties (Annals of Mathematical Studies vol 131)* (Princeton, NJ: Princeton University Press)

[40] Leung N C and Vafa C 1998 *Adv. Theor. Math. Phys.* **2** 91 (*Preprint* hep-th/9711013)

[41] Cox D 1995 The homogeneous coordinate ring of a toric variety *J. Algebr. Geom.* **4** 17

[42] Kreuzer M and Skarke H 2000 Reflexive polyhedra, weights and toric Calabi–Yau fibrations *Preprint* math.AG/0001106

Avram A C, Kreuzer M, Mandelberg M and Skarke H 1997 The web of Calabi–Yau hypersurfaces in toric varieties *Nucl. Phys.* B **505** 625–40 (*Preprint* hep-th/9703003)

Kreuzer M and Skarke H 1998 Calabi–Yau 4-folds and toric fibrations *J. Geom. Phys.* **26** 272–90 (*Preprint* hep-th/9701175)

Anselmo F, Ellis J, Nanopoulos D V and Volkov G 2002 Universal Calabi–Yau algebra: towards an unification of complex geometry *Preprint* hep-th/0207188

[43] Witten E 1993 *Nucl. Phys.* B **403** 159–22 (*Preprint* hep-th/9301042)

[44] Belhaj A and Saidi E H 2001 Hyper-Kahler singularities in superstrings compactification and 2d $N = 4$ conformal field theory *Class. Quantum Grav.* **18** 57–82 (*Preprint* hep-th/0002205)

Belhaj A and Saidi E H 2000 On hyper-Kahler singularities *Mod. Phys. Lett.* A **15** 1767–79 (*Preprint* hep-th/0007143)

[45] Hori K and Vafa C 2000 Mirror symmetry *Preprint* hep-th/0002222

[46] Hori K, Iqbal A and Vafa C 2000 D-branes and mirror symmetry *Preprint* hep-th/0005247

[47] Aganagic M and Vafa C 2000 Mirror symmetry, D-branes and counting holomorphic discs *Preprint* hep-th/0012041

Aganagic M and Vafa C 2001 Mirror symmetry and a G_2 flop *Preprint* hep-th/0105225

Aganagic M, Klemm A and Vafa C 2001 Disk instantons, mirror symmetry and the duality web *Preprint* hep-th/0105045

[48] Belhaj A 2002 Mirror symmetry and Landau–Ginzburg Calabi–Yau superpotentials in F-theory compactifications *J. Phys. A: Math. Gen.* **35** 965–84 (*Preprint* hep-th/0112005)

[49] Batyrev V V and Materov E N 2002 Toric residues and mirror symmetry *Preprint* math.AG/0203216

[50] Batyrev V V 1997 Toric degenerations of Fano varieties and constructing mirror manifolds *Preprint* alg-geom/9712034

Batyrev V V 1994 Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varities *J. Algebra. Geom.* **3** 493

Batyrev V V 1994 *Duke Math. J.* **75** 293

[51] Cachazo F A and Vafa C 2000 Type I and real algebraic geometry *Preprint* hep-th/0001029

[52] Horava P and Witten E 1996 Eleven-dimensional supergravity on a manifold with boundary *Nucl. Phys.* B **475** 94–114 (*Preprint* hep-th/9603142)

[53] Blumenhagen R, Braun V, Kors B and Lust D 2002 Orientifolds of $K3$ and Calabi–Yau manifolds with intersecting D-branes *Preprint* hep-th/0206038